Patterns in Probability (Extended) Solutions

In the following, W stands for a point won by me, L for a point won by my opponent

1. $a + b = 1$

2. p(I win first 2 points) = $a \times a = a^2$ (the events are independent, so we simply multiply individual probabilities)

3. For this to happen, the points sequence must go: LWW

$\Rightarrow$ p(I win a 3-point game) = a^2b

4. For this to happen, the points sequence must go: WLWW

$\Rightarrow$ p(I win a 4-point game) = a^3b

Extending this (so we have enough data to formulate the probability I win an N-point game):

To win a 5-point game, the points sequence must go: LWLWW

$\Rightarrow$ *p(I win a 5-point game) =* a^3b^2

To win a 6-point game, the points sequence must go: WLWLWW

$\Rightarrow$ *p(I win a 6-point game) =* a^4b^2

To win a 7-point game, the points sequence must go: LWLWLWW

$\Rightarrow$ *p(I win a 7-point game) =* a^4b^3

To win a 8-point game, the points sequence must go: WLWLWLWW

$\Rightarrow$ *p(I win a 5-point game) =* a^5b^3

5. There are a number of patterns on display, such as:

- The total of the powers of a and b is equal to the number of points played
- For me to win an even-point game, I must win the first point
- For me to win an odd-point game, I must lose the first point
- When I win an even-point game I outscore my opponent by 2 points
- When I win an odd-point game I outscore my opponent by 1 point

There are actually two sets of patterns – one for when N is even, and one for when N is odd (see answer to Q6 below).

6. To win a N-point game, it depends if N is even or odd:

N even $\Rightarrow$ p(I win a N-point game) = $a^{\frac{N}{2}+1}b^{\frac{N}{2}-1}$

N odd $\Rightarrow$ p(I win a N-point game) = $a^{\frac{N+1}{2}}b^{\frac{N-1}{2}}$

[Note: $N \geq 2$]

7. The probability I win a game in 5 points or less is:

= p(I win in 2 points) + p(I win in 3) + p(I win in 4) + p(I win in 5)

= $a^2 + a^2b + a^3b + a^3b^2$

If a = 0.6 and b = 0.4, this gives a probability of $(0.6)^2 + (0.6)^2(0.4) + (0.6)^3(0.4) + (0.6)^3(0.6)^2$

which is 0.62496 which is 0.625 (3sf)

8. Proof of answer(s) to question 6:
In the case when N is even, I need to win 2 more points than my opponent.
This means that if I win x points, my opponent wins N – x, and so x – (N – x) = 2.

$\Rightarrow \; 2x - N = 2 \;\Rightarrow\; x = \frac{N+2}{2} = \frac{N}{2} + 1$ and $N - x = N - (\frac{N}{2} + 1) = \frac{N}{2} - 1$

These are independent events, so

p(I win) = $a^x b^{N-x} = a^{\frac{N}{2}+1}b^{\frac{N}{2}-1}$

In the case when N is odd, I need to win 1 more point than my opponent.
Again, if I win x points, my opponent wins N – x, and so x – (N – x) = 1.

$\Rightarrow \; 2x - N = 1 \;\Rightarrow\; x = \frac{N+1}{2}$ and $N - x = N - \frac{(N+1)}{2} = \frac{N-1}{2}$

These are still independent events, so

p(I win) = $a^x b^{N-x} = a^{\frac{N+1}{2}}b^{\frac{N-1}{2}}$

Another possible justification
If my opponent and I are equally matched, then a = b = 0.5

N even $\Rightarrow$ p(I win a N-point game) = $a^{\frac{N}{2}+1}b^{\frac{N}{2}-1} = a^{\frac{N}{2}+1}a^{\frac{N}{2}-1} = a^N = 0.5^N$

N odd $\Rightarrow$ p(I win a N-point game) = $a^{\frac{N+1}{2}}b^{\frac{N-1}{2}} = a^{\frac{N+1}{2}}a^{\frac{N-1}{2}} = a^N = 0.5^N$

Also, if I we are evenly matched, the above are the probabilities that my opponent wins

So, probability I win in 2 or 3 or 4 or …. games is:

$(0.5)^2 + (0.5)^3 + (0.5)^4 + (0.5)^5 + \ldots\ldots = 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + \ldots\ldots$ which tends to 0.5

And the probability my opponent wins in 1 or 2 or 3 or 4 or …. games is the same